Study on the Statistical Properties of Stochastic Optimization Problems

Summer Internship at the Statistical Laboratory of the University of Cambridge
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Abstract

Causal statistics often necessitate sensitivity analysis to account for uncertainties and unmeasured confounders in estimating causal effects. Recently, a novel optimization-based sensitivity analysis framework was introduced, relying on solving two stochastic optimization problems where random variables are replaced by data-derived estimators. This framework highlights the importance of understanding the consistency properties of the optimal values of the problems, particularly as they depend on the consistency of the estimators.

In this report, we present an overview of the problem, existing theoretical results, and the analysis conducted during the internship. We developed a methodology to examine the consistency properties of the optimal values of these stochastic optimization problems and identified conditions under which consistency is guaranteed in the specific context of the sensitivity analysis framework. This work offers a foundational example that could inspire broader investigations into consistency properties in stochastic optimization.

Introduction

In the following report we summarize the work done during a summer internship conducted at the Statistical Laboratory of the University of Cambridge along a visiting period of 5 weeks. The project was built upon the paper of Freidling and Zhao, *Optimization-based Sensitivity Analysis for Unmeasured Confounding using Partial Correlations*, [2] and this report made use of the works in bibliography as well as some manuscript notes written by Tobias Freidling, who contributed directly to the work.

Motivations

Causal inference is a cornerstone of modern statistics, focusing on the estimation of cause-and-effect relationships between variables. Unlike traditional statistical approaches, which are often limited to uncovering correlations, causal inference aims to uncover the mechanisms driving these associations. The ability to accurately estimate causal effects is critical across a range of disciplines, from medicine and biology to economics and social sciences, where it informs evidence-based decision-making and policy design.

However, establishing causal relationships is fraught with challenges. One of the most significant is the problem of unmeasured confounders—hidden variables that influence both the treatment and the outcome, potentially biasing estimates of causal effects. For instance, in observational studies, it is rarely possible to measure every factor that could influence both a treatment (e.g., a drug) and its outcome (e.g., patient recovery). This introduces uncertainty and necessitates additional methods to account for the confounding effect of these hidden variables.

A common approach to address unmeasured confounding is the use of randomized controlled trials (RCTs), where subjects are randomly assigned to treatment and control groups. This randomization ensures that confounders are evenly distributed across groups, mitigating their effect. While powerful, RCTs have significant limitations: they can be costly, time-consuming, ethically questionable in certain scenarios, and often impractical for large-scale or complex systems. Thus, while RCTs are considered the gold standard, they are not always feasible or efficient.

To address the challenges of observational data and unmeasured confounders, sensitivity analysis has emerged as a critical tool in causal inference. Sensitivity analysis measures the robustness of causal estimates to potential violations of assumptions, such as the presence of hidden confounders. By quantifying how sensitive a causal conclusion is to these unmeasured factors, researchers can better understand the reliability of their findings.

Recently, a novel optimization-based sensitivity analysis framework was proposed by Freidling and Zhao in [2], which introduces a new paradigm for addressing unmeasured confounding. This framework leverages stochastic optimization techniques to bound the causal effect of interest. The methodology works by formulating two optimization problems, representing the minimum and maximum possible values of the causal effect under a set of plausible assumptions about the unmeasured confounder. By solving these problems, the framework defines a range—known as the partially identified region—within which the true causal effect likely lies. This approach provides a more structured and quantitative alternative to traditional sensitivity analyses, offering actionable insights even in the presence of significant uncertainty.

Contributions

This report builds upon the optimization-based sensitivity analysis framework by investigating the consistency properties of the stochastic optimization problems involved. Specifically, the focus of this study is on understanding the conditions under which the optimal values of these problems are consistent with the underlying data estimators. This consistency analysis is crucial for validating the reliability of the framework's sensitivity measures and extending its applicability.

The work presented here includes a detailed review of the theoretical background, a methodology for examining consistency properties, and the derivation of sufficient conditions for consistency in the context of sensitivity analysis. By combining statistical and optimization perspectives, this study contributes a foundational example for further research into consistency in stochastic optimization.

The structure of this report is as follows: chapter 1 provides the statistical and optimization background necessary to understand the problem; chapter 2 presents the core analysis, including the results and insights obtained during the project; chapter 3 concludes the report with a summary of findings and directions for future work.

Chapter 1

Background

1.1 Sensitivity Analysis Framework

1.1.1 Causal Problem

The problems of statistical study of stochastic optimization problems we addressed arises from a paper of Freidling and Zhao [2]. In the paper, the authors proposed a sensitivity analysis framework based on a stochastic optimization problem. In this section we will present the framework and its construction and discuss the optimization problem it arises.

We consider a setting where we have a sample $(U_i, V_i)_{i=1}^n$ drawn i.i.d. from a population with distribution $\mathbb{P}_{V,U} = \mathbb{P}$. We denote with V the variables observed and we indicate with $\mathbb{P}_V$ its marginal distribution, while U represents an unmeasured confounder. The main purpose is to measure the causal effect of an observed treatment variable D to an observed outcome Y; the non triviality of this problem lies in the fact that the confounder, for definition, influences both the treatment and the outcome, and is unobserved. The basic model is therefore composed of the observed $V = (Y, D, X)$ where D is a treatment variable, Y is the outcome variable and X is covariate, i.e. an independent variable that can influence the outcome but not of direct interest. The variables are then connected in causal relationships as shown in figure 1.1.

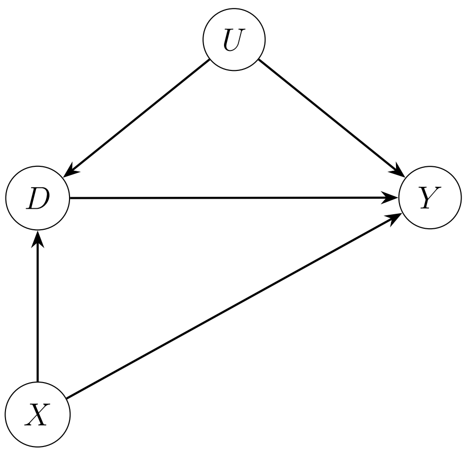

Figure 1.1: Causal diagram of the relationships between U, D, Y, and X.

The causal effect we would like to study is the one of the treatment D on the outcome Y, and thus try to eliminate the effect of covariate X and confounder U.

Our model can be made more accurate by including also an observed instrumental variable Z for D. We refer to Z as a valid instrument for D if:

(i) Z is an independent predictor of D, i.e. $R_{Z\sim D|X} \neq 0$;

(ii) Z is partially uncorrelated with the unmeasured confounder U, i.e. $R_{Z\sim U|X} = 0$;

(iii) Z has no influence on the outcome Y not mediated by D, i.e. $R_{Y\sim Z|X,U,D} = 0$,

We observe that condition (i) depends only on the observed data V and can therefore be directly verified. Conditions (ii) and (iii), on the other hand, rely also on the unmeasured confounder U, so they require sensitivity analysis, which means that we must verify if the conditions are robust to variations of U.

The resulting model, that is the one we will refer to in our analysis, comprehends as observed variable $V = (Y, D, Z, X)$ and has causal diagram represented in figure 1.2.

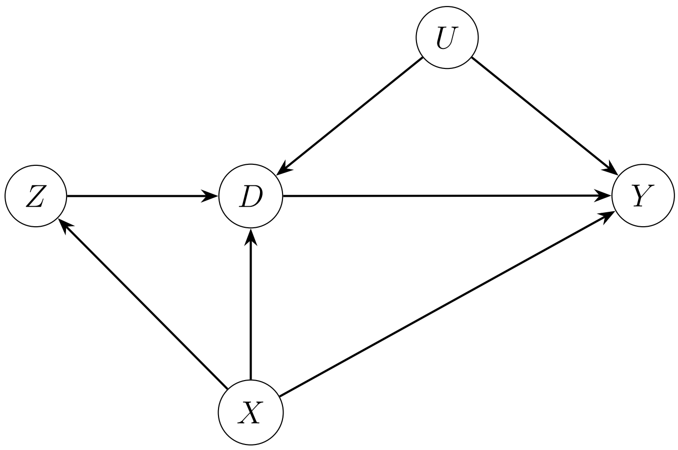

Figure 1.2: A causal diagram with an additional variable Z affecting D and influenced by X.

In the figure, note that condition (i) of Z being a valid instrument is represented in the edge going from Z to D, while the absence of edges from U to Z and from Z to Y that express conditions (ii) and (iii).

1.1.2 Graph Interpretation

In the paper [2], the aim is to produce a sensitivity measure of the causal effect of D on Y, which eliminates the dependence by the other observed variables X and Z and by the unmeasured confounder U. This is done by conducting inference on a functional $\beta = \beta(\mathbb{P}_{V,U})$ which measures the causal effect of interest.

To obtain a measurable quantity from the causal model described in the section before, we can translate the causal diagram of figure 1.2 in algebraic relations, using the matrix algebra for graphical statistical models described in [4].

In particular, a causal graph corresponds to a linear system, which it means that the variables $(V, U) = (Y, D, Z, X, U)$ satisfy the relation

$$\begin{pmatrix} V \\ U \end{pmatrix} = b^t \begin{pmatrix} V \\ U \end{pmatrix} + E \tag{1.1}$$

where b is a vector of coefficients and E is a random gaussian vector of mean 0 and covariance matrix Λ which represents a random noise. b and Λ are determined using a weight function σ relative to the graph, whose properties are more thoroughly described in [4].

To obtain b and E, the weight function is applied respectively to $W[V \longrightarrow V]$, the matrix of one-directional edges, where every entry is the set of edges from two corresponding random variables, and $W[Y \longleftrightarrow V]$, the matrix of bidirectional edges. In formula:

$$b = \sigma(W[V \longrightarrow V]) \tag{1.2}$$

$$\Lambda = \sigma(W[Y \longleftrightarrow V]) \tag{1.3}$$

For our model the matrix of edges, taking the entries in the order Y, D, Z, X, U, are the following:

$$W[V \longrightarrow V] = \begin{pmatrix} \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\ \{D \to Y\} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \{Z \to D\} & \varnothing & \varnothing & \varnothing \\ \{X \to Y\} & \{X \to D\} & \{X \to Z\} & \varnothing & \varnothing \\ \{U \to Y\} & \{U \to D\} & \varnothing & \varnothing & \varnothing \end{pmatrix} \tag{1.4}$$

For the bidirectional edges matrix, we consider a loop for each variable, so we obtain the following:

$$W[Y \longleftrightarrow V] = \begin{pmatrix} \{Y \to Y\} & \varnothing & \varnothing & \varnothing & \varnothing \\ \varnothing & \{D \to D\} & \varnothing & \varnothing & \varnothing \\ \varnothing & \varnothing & \{Z \to Z\} & \varnothing & \varnothing \\ \varnothing & \varnothing & \varnothing & \{X \to X\} & \varnothing \\ \varnothing & \varnothing & \varnothing & \varnothing & \{U \to U\} \end{pmatrix} \tag{1.5}$$

The weight function σ works entry-wise assigning a weight to each walk and we then use them to define the system as in (1.1). We therefore obtain a system of equations for Y, D, Z, X, U. The ones we are interested in are Y and D, since we are studying the causal effect of the latter on the former. The corresponding equations will be of the form

$$Y = b_{D,Y} D + b_{U,Y} U + b_{X,Y} X + E_Y \tag{1.6}$$

$$D = b_{Z,D} Z + b_{X,D} X + b_{U,D} U + E_U \tag{1.7}$$

where E_Y and E_U are two random variables normally distributed with mean 0 and variance respectively $\lambda_{Y,Y}$ and $\lambda_{U,U}$.

We are interested in measuring the causal effect of D on Y. This is done by determining the coefficient of D in the decomposition (1.6), once excluded the effect of the covariate X and of the confounder U. We can obtain the component of Y which does not depend on X and U by computing the residual of Y after regressing out X and U.

In general, we define the residual of a random variable S after regressing out another random variable T, as

$$S^{\perp T} := S - T^t \operatorname{var}(T)^{-1} \operatorname{cov}(S, T) \tag{1.8}$$

We can interpret this residual as the coefficient of the regression of S on T; therefore the residual $S^{\perp T}$ represents S without the linear effect of T on S. Since we are considering linear dependence of variables connected by a causal relation, eliminating the linear effect corresponds to eliminating the effect of the variables.

Reversing the formula 1.8, we obtain the following decomposition of the random variable S:

$$S = S^{\perp T} + T^t Var(T)^{-1} cov(S,T) \tag{1.9}$$

In this formulation, it becomes clear that $\mathrm{var}(T)^{-1}\,\mathrm{cov}(S,T)$ is the coefficient of U in the decomposition of S in the direction parallel to T.

Applying the decomposition in our case from equations 1.6 and 1.7, we can write:

$$D = D^{\perp U,X} + b_{U,D}U + b_{X,D}X \tag{1.10}$$

And thus

$$Y = Y^{\perp U,X} + b_{U,Y}U + b_{X,Y}X \tag{1.11}$$

where $D^{\perp X,U}$ is included in the term $Y^{\perp U,X}$.

For the definition given in 1.8, we have that $\beta := \mathrm{var}(D^{\perp X,U})^{-1}\,\mathrm{cov}(Y^{\perp X,U}, D^{\perp X,U})$ is the coefficient of $D^{\perp X,U}$ in the decomposition of $Y^{\perp X,U}$, i.e.:

$$Y^{\perp X,U} = \beta D^{\perp X,U} + \tilde{b}_X X + \tilde{b}_U U \tag{1.12}$$

Thus, combining 1.12, 1.10 and 1.6 we have that

$$Y = \beta D + b_{X,Y}X + b_{U,Y}U \tag{1.13}$$

Comparing 1.6 and 1.13, it becomes clear that

$$b_{D,Y} = \beta = \frac{Cov(Y^{\perp X,U}, D^{\perp X,U})}{Var(D^{\perp X,U})}$$

Since the coefficient in the linear model corresponds to the causal effect, we have that so-defined β is the sensitivity measure we are looking for. We also note that $\beta = \beta(\mathbb{P}_{V,U})$, i.e. β depends on $\mathbb{P}_{V,U}$.

1.1.3 R-values

The idea to build the sensitivity model is then to rewrite β in terms of R- or partial R-values, which are more tractable objects.

First of all, we state the corresponding definitions, that make use of the residual defined in the precedent section:

Definition 1.

$$R_{Y\sim X} := \sqrt{1 - \frac{\mathrm{var}(Y^{\perp X})}{\mathrm{var}(Y)}} \tag{1.14}$$

$$R_{Y\sim X|Z} := \mathrm{corr}(Y^{\perp Z}, X^{\perp Z}) \tag{1.15}$$

We also define the R^2 and partial R^2-values

Definition 2.

$$R^2_{Y\sim X} := 1 - \frac{\operatorname{var}(Y^{\perp X})}{\operatorname{var}(Y)} \tag{1.16}$$

$$R^2_{Y\sim X|Z} := \frac{R^2_{Y\sim X+Z} - R^2_{Y\sim X}}{1 - R^2_{Y\sim Z}} \tag{1.17}$$

We easily see that $R^2_{Y\sim X} = (R_{Y\sim X})^2$. The relationship between the partial R- and R^2-values comes from the following:

Proposition 3. *If X is a one-dimensional random variable, it follows that $R^2_{Y\sim X|Z} = (R_{Y\sim X|Z})^2$.*

The R and partial R-values can be written in terms of the covariance matrix Σ, i.e. given $X_1, \ldots, X_p$ random variables, the matrix for which $(\Sigma)_{i,j} := \operatorname{cov}(X_i, X_j)$ if $i \neq j$ and $(\Sigma)_{i,i} := \operatorname{var}(X_i)$.

To this regard we can easily show that

$$R_{Y\sim X} = \operatorname{corr}(X,Y) := \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}} \tag{1.18}$$

The argument is the following:

$$\begin{aligned}
R_{Y\sim X} &= \sqrt{\frac{\operatorname{var}(Y) - \operatorname{var}(Y^{\perp X})}{\operatorname{var}(Y)}} \\
&= \sqrt{\frac{\operatorname{var}(Y) - \operatorname{var}(Y + X^t \operatorname{var}(X)^{-1}\operatorname{cov}(X,Y))}{\operatorname{var}(Y)}} \\
&= \sqrt{\frac{\operatorname{var}(Y) - \operatorname{var}(Y) - \operatorname{var}(X)\operatorname{var}(X)^{-2}\operatorname{cov}(X,Y)^2}{\operatorname{var}(Y)}} \\
&= \frac{\operatorname{cov}(X,Y)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Y)}}
\end{aligned}$$

If we have a collection of variables $X_1, X_2, \ldots, X_p$, we can therefore write $R_{X_i\sim X_j} = (\Sigma)_{i,j}$.

We can rewrite also the partial R-values in terms of the covariance matrix Σ with some more effort: given $X_1, \ldots, X_p$, we consider $R_{X_i\sim X_j|X_{I\setminus\{i,j\}}}$, with $I \subseteq [p] := \{1, \ldots, p\}$ and $i, j \in I$.

We then define

$$\Omega^I_{ij} := (E_I e_i)^T (E_I^T \Sigma(\psi) E_I)^{-1} (E_I e_j), \tag{1.19}$$

where $e_k \in \mathbb{R}^p$ has 1 at its k-th component and 0 everywhere else; $E_I \in \mathbb{R}^{p\times|I|}$ is the matrix that selects the columns corresponding to I, for instance if $I = \{1, 2, 3\}$, then

$$E_I = \begin{pmatrix} I_{3\times 3} \\ 0 \end{pmatrix}. \tag{1.20}$$

So, to get Ω^I_{ij}, we first subset the matrix Σ, then invert it and then take the entry that corresponds to (i, j) in the full matrix. The partial correlation of X_i and X_j given $X_{I\setminus\{i,j\}}$

can be expressed as

$$g_{ij}^{I}(\psi) = R_{X_i \sim X_j | X_{I\setminus\{i,j\}}} = -\frac{\Omega_{ij}^{I}}{\sqrt{\Omega_{ii}^{I}\,\Omega_{jj}^{I}}}. \tag{1.21}$$

To go on with of our sensitivity model, we can rewrite β in terms of R- and partial R-values: in this way we can distinguish the terms that do not depend on U and that we can summarize in a factor $\theta(\mathbb{P}_V)$ and the terms which depend also on U in the factor $\psi(\mathbb{P}(U, V))$. In the paper [2] there are shown the details of how to write β in terms of the R-values. For our scope the important thing to note is that we can rewrite β as:

$$\beta = \beta(\psi(\mathbb{P}_{U,V}), \theta(\mathbb{P}_V)) \tag{1.22}$$

where the parameter ψ is function of R- and partial R-values which contain the unknown confounder U.

1.1.4 Optimization Problem

Our aim is then to conduct inference on the parameter $\beta = \beta(\psi(\mathbb{P}_{U,V}), \theta(\mathbb{P}_V))$. Obviously we can estimate observable term $\theta(\mathbb{P}_V)$, but not on the unobserved factor included in $\psi(\mathbb{P}_{U,V})$.

We make the assumption to be in a *partially identified* setting, i.e.

$$\psi \in \Psi(\theta)$$

where as $\Psi(\theta)$ is a set depending on θ.

We are restricting to a range of possible values for the unobserved term ψ. Therefore, we can define a region of possible values of β for corresponding values of ψ. This is called *partially identified region* (PIR) and is defined as

$$PIR(\mathbb{P}_V) := \{\beta(\theta(\mathbb{P}_V), \psi) : \psi \in \Psi(\theta(\mathbb{P}_V))\}$$

We also assume that $\beta \in \mathbb{R}$. In this way we can bound the sensitivity parameter by the optimal values of the two following optimization problems:

$$\text{min/max} \quad \beta(\theta(\mathbb{P}_V), \psi) \tag{1.23}$$

$$\text{subject to} \quad \psi \in \Psi(\theta(\mathbb{P}_V)) \tag{1.24}$$

To define the optimization problem we will write explicitly the constraints that determine $\Psi(\theta)$, that will also be the object of our analysis. In particular, as we said, ψ is function of R- and partial R-values which include the unknown confounder U. The constraints are thus defined in term of inequalities of them. We will consider the constraints in table 1.1, taken from table 1 in the paper [2], based on corresponding edges in the causal diagram.

As we see the constraints are inequality constraints on partial R-values or relation constraints on partial R^2-values. We note also that in the specification of the constraints we introduced the variables $\tilde{X}$ and $\dot{X}$. These are a decomposition of the random vector $X = (\dot{X}, \tilde{X})$, where $\dot{X}$ indicates the variables in X that do not have influence on U, while with $\tilde{X}$, we indicate variables in X that can have influence on U. In terms of causal diagram, this leads to a situation as in figure 1.3.

Table 1.1: Specification of the Sensitivity Model

Edge	Sensitivity Bound
$U \to D$	1. $R_{D\sim U\|X,Z} \in [B^l_{\mathrm{UD}}, B^u_{\mathrm{UD}}]$
	2. $R^2_{D\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UD} R^2_{D\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z}$
$U \to Y$	1. $R_{Y\sim U\|X,Z,D} \in [B^l_{\mathrm{UY}}, B^u_{\mathrm{UY}}]$
	2. $R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UY} R^2_{Y\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z}$
	3. $R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z,D} \leq b_{UY} R^2_{Y\sim \dot{X}_J\|\tilde{X},\dot{X}_I,Z,D}$
$U \leftrightarrow Z$	1. $R_{Z\sim U\|X} \in [B^l_{\mathrm{UZ}}, B^u_{\mathrm{UZ}}]$
	2. $R^2_{Z\sim U\|\tilde{X},\dot{X}_{-j}} \leq b_{UZ} R^2_{Z\sim \dot{X}_j\|\tilde{X},\dot{X}_{-j}}$
$Z \to Y$	1. $R_{Y\sim Z\|X,U,D} \in [B^l_{\mathrm{ZY}}, B^u_{\mathrm{ZY}}]$
	2. $R^2_{Y\sim Z\|X,U,D} \leq b_{ZY} R^2_{Y\sim \dot{X}_j\|\tilde{X},\dot{X}_{-j},Z,U,D}$

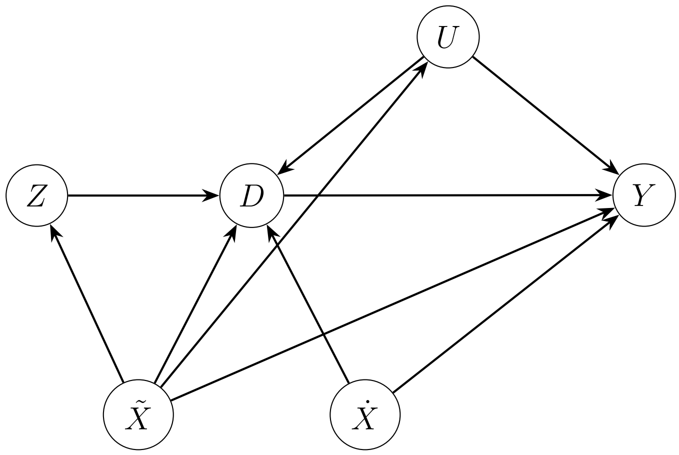

Figure 1.3: A causal diagram with $\tilde{X}$ influencing D, U, and Y, and $\dot{X}$ influencing D and Y.

In terms of R-values, this translates to the constraint:

$$R^2_{U\sim \dot{X}|\tilde{X},Z} = 0 \tag{1.25}$$

We will also let only the variables in $\dot{X}$ to be in front of the conditioning in the R^2-values constraints.

Once defined the constraints of the optimization problem, we conduct inference on the sensitivity parameter β by including the data from the observations of V in an estimator $\hat{\theta}$ of the parameter $\theta(\mathbb{P}_V)$. We plug-in the estimator $\hat{\theta}$ and then solve the resulting optimization problem:

$$\begin{aligned} \text{min/max} \quad & \beta(\hat{\theta}, \psi) \\ \text{subject to} \quad & \psi \in \Psi(\hat{\theta}) \end{aligned}$$

If the estimator $\hat{\theta}$ is consistent we have that $\hat{\theta} \to \theta$ for the number of samples $n \longrightarrow \infty$

The real distribution of the statistic $\hat{\theta}$, given the observed data, is then approximated through a bootstrap approach. We produce a collection of bootstrapped estimators $\hat{\hat{\theta}}$ and then solve the optimization problem for each of them finding the optimal value $\hat{\hat{\beta}}$. We have consistency of the bootstrap approximation, in the sense that the distribution of $\hat{\hat{\theta}}$ converges to $\hat{\theta}$ growing the number of bootstraps.

In both cases the question that naturally arises is the consistency of the optimal values obtained, i.e. whether we can say that the optimal values of the approximate problems, $\hat{\beta}$ and $\hat{\hat{\beta}}$, converge to the real optimal value β of 1.23.

In general the problem that arises is: knowing that we have the convergence of a statistic $t_n \longrightarrow t$ (in some sense), called β_n the optimal value of the optimization problem

$$\begin{array}{rl} \min & \beta(t_n, \psi) \\ \text{s.t.} & \psi \in \Psi(t_n) \end{array} \tag{1.26}$$

and β the optimal value of

$$\begin{array}{rl} \min & \beta(t, \psi) \\ \text{s.t.} & \psi \in \Psi(t) \end{array} \tag{1.27}$$

under which hypothesis does $\beta_n \longrightarrow \beta$ (in some sense)? And, in particular, does it converge in our sensitivity analysis setting?

1.2 Optimization Results

To address the consistency problem we can rely on some relevant results from optimization theory and try to verify the hypothesis under which consistency of the optimal value holds. The results are contained in the work [3] and in the book [1].

In this section we will consider a general stochastic optimization problem, in the form:

$$\begin{array}{llll} \nu(u) = \min\limits_{x \in \mathbb{R}^k} f(x, u) & \text{subject to} & g_i(x, u) = 0, & i = 1, \ldots, q; \\ & & g_i(x, u) \leq 0, & i = q+1, \ldots, p, \end{array}$$

where u will represent an element in $\mathbb{R}^{k'}$ and we will assume that f and the g_i are smooth enough (or convex).

In order to study consistency of the optimal value, we will consider the approximate problem substituting u with $\hat{u}_n$:

$$\begin{array}{llll} \nu(\hat{u}_n) = \min\limits_{x \in \mathbb{R}^k} f(x, \hat{u}_n) & \text{subject to} & g_i(x, \hat{u}_n) = 0, & i = 1, \ldots, q; \\ & & g_i(x, \hat{u}_n) \leq 0, & i = q+1, \ldots, p, \end{array}$$

where $\hat{u}_n$ is a consistent estimator of u. We can consider the approximated problem of a sample average approximation of the real one, i.e. $\hat{u}_n = \frac{1}{n} \sum_{i=1}^n u_i$ with u_i i.i.d. samples of u, to take advantage of the sample average approximation.

1.2.1 Consistency Results

We would like to find the hypothesis under which consistency results hold. To do so, we first introduce some definitions:

Definition 4. Let X and Y be vector (linear) normed spaces and $g \colon X \to Y$.

The function g is *directionally differentiable* at $x \in X$ in a direction $h \in X$, if for any sequence $t_n \downarrow 0$, the limit

$$g'(x,h) := \lim_{n\to\infty} \frac{g(x+t_n h) - g(x)}{t_n}$$

exists. If the directional derivative exists in every direction and is linear and continuous in h, g is called *Gâteaux differentiable.*

The function g is *Hadamard directionally differentiable* at $x \in X$ in a direction $h \in X$, if for any sequences $h_n \to h$ and $t_n \downarrow 0$, the limit

$$g'(x,h) := \lim_{n\to\infty} \frac{g(x+t_n h_n) - g(x)}{t_n}$$

exists. If the directional derivative exists in every direction and is linear in h, g is called *Hadamard differentiable.* (g is automatically continuous.)

The function g is *Fréchet directionally differentiable* at x, if it is directionally differentiable at x and

$$g(x+h) = g(x) + g'(x,h) + o(\|h\|), \quad h \in X.$$

If $g'(x,\cdot)$ is also linear and continuous, g is called *Fréchet differentiable.*

Clearly, Hadamard (dir.) differentiability or Fréchet (dir.) differentiability imply Gâteaux (dir.) differentiability. If X is finite dimensional, Hadamard directional differentiability implies Fréchet directional differentiability. Moreover, if g is Fréchet directionally differentiable and $g'(x,\cdot)$ is continuous, then g is also Hadamard directional differentiable. In particular, Fréchet differentiability implies Hadamard (directional) differentiability.

If a function is not directionally differentiable, we may still be able to define upper and lower "bounds".

Definition 5. Let $f\colon X \to \bar{\mathbb{R}}$ be an extended real valued function and $x \in X$ such that $f(x)$ is finite. The upper and lower directional derivatives of f at x in direction h are defined as

$$f'_+(x,h) := \limsup_{t\downarrow 0} \frac{f(x+th) - f(x)}{t}$$

$$f'_-(x,h) := \liminf_{t\downarrow 0} \frac{f(x+th) - f(x)}{t}$$

The function f is directionally differentiable at x in direction h if $f'_+(x,h)$ and $f'_-(x,h)$ are equal. We can define upper and lower Hadamard directional derivatives analogously.

Definition 6. Let X be a Banach space, $S \subset X$ and $x \in S$. Then, the *contingent (Bouligand)/tangent cone* is given by

$$T_S(x) = \{h \in X \colon \exists\, t_n \downarrow 0 \colon \operatorname{dist}(x+t_n h, S) = o(t_n)\} = \{h \in X \colon \exists\, t_n \downarrow 0, \exists\, h_n \to h \colon x + t_n h_n \in S\,\forall\, n \in \mathbb{N}$$

We will also use the following notion of convergence:

Definition 7. Given $\{X_N\}_N, X$ random elements in a Banach space B, we say, when it makes sense, that X_N *converges weakly to* X, $X_N \implies X$ as $N \to \infty$, if

$$\mathbb{E}[f(X_N)] \to \mathbb{E}[f(X)] \tag{1.28}$$

as $N \to \infty$, for every $f : B \to \mathbb{R}$ bounded and continuous.

To guarantee consistency we will then use mainly two different results. The first is the continuous mapping theorem and is stated as follows:

Theorem 8. *Let S and S' metric spaces and $\{X_n\}_{n\geq 1}$ random elements on S. Suppose $g : S \to S'$ continuous a.s., then*

$$i \; X_n \xrightarrow{d} X \quad \Rightarrow \quad g(X_n) \xrightarrow{d} g(X);$$

$$ii \; X_n \xrightarrow{\mathbb{P}} X \quad \Rightarrow \quad g(X_n) \xrightarrow{\mathbb{P}} g(X);$$

$$iii \; X_n \xrightarrow{a.s.} X \quad \Rightarrow \quad g(X_n) \xrightarrow{a.s.} g(X).$$

It is clear that a possibility to show consistency is to use this theorem with ν as g and $\hat{u}_n, u$ as X_n, X. Therefore, proving that ν is continuous will guarantee that the convergence of the estimator implies convergence of the optimal value.

Another important result to prove consistency is the following theorem, known as *Delta method*.

Theorem 9. *Let B_1 and B_2 Banach spaces, equipped with their Borel σ-algebras, $\{Y_n\}_{n\geq 1}$ a sequence of random elements of B_1, $g : B_1 \to B_2$ a mapping and τ_n a sequence of positive numbers tending to infinity. Suppose that B_1 is separable, g is Hadamard directionally differentiable at a point $\mu \in B_1$, and that $X_n := \tau_n[Y_n - \mu]$ converges weakly to a random element Y of B_1. Then*

$$\tau_n[g(Y_n) - g(\mu)] \implies g'_\mu(Y)$$

and

$$\tau_n[g(Y_n) - g(\mu)] = g'_\mu(X_n) + o_{\mathbb{P}}(1)$$

The idea is to apply the Delta theorem to our optimization problem, taking as Y_n the optimal solution $\hat{u}_n$ of the approximated problem and as G the function ν which gives the optimum value of our stochastic problem. We therefore need to choose τ_n and μ as in the theorem, such that $\tau_n(\hat{u}_n - \mu)$ converges in distribution. To to so we can take advantage of the central limit theorem in the following formulation.

Theorem 10. *Let $\{X_n\}_{n\geq 1} \subset \mathcal{L}^2$ i.i.d. with $\mu = \mathbb{E}[X_1]$ and $\sigma^2 = Var(X_1) \geq 0$, then*

$$\frac{S_n/n - \mu}{\sqrt{n}} := \frac{\frac{X_1+X_2+\cdots+X_n}{n} - \mu}{\sqrt{n}} \implies Y \sim \mathcal{N}(0, \sigma^2) \tag{1.29}$$

Therefore, by choosing $\tau_n = \sqrt{n}$ and $\mu = u$, we have for CLT that $\sqrt{n}(\hat{u}_n - u) \implies Z \sim \mathcal{N}(0, \sigma^2)$. If we have that ν is Hadamard differentiable, the Delta theorem, gives us the consistency we would like to prove, namely

$$\sqrt{n}(\nu(\hat{u}_n) - \nu(u)) \implies \nu'_u(Z) \tag{1.30}$$

The main problem is therefore to prove Hadamard differentiability for the function ν which maps the optimal value to the parameter of the optimization problem.

1.2.2 Constraint Qualifications

In order to define sufficient condition for continuity and Hadamard differentiability for the optimal value function $\nu(\cdot)$ of the problem we introduce the concept of constraint qualification.

Constraint qualifications (CQ) are regularity conditions for the *analytic* description of sets. In that sense, they assess how well inequalities and equations describe a geometric object. There are two reasons why constraint qualifications can be violated: (1) the set we try to describe is very complicated, (2) our description is bad.

For instance, we can describe the non-negative real numbers $[0, \infty)$, via the inequalities $\{x \in \mathbb{R} \colon x \geq 0\}$, $\{x \in \mathbb{R} \colon x^3 \geq 0\}$, or even $\{x \in \mathbb{R} \colon x \geq 0, x^3 \geq 0\}$. The last description is obviously redundant (and in that sense bad); for more complex sets, however, it may not be that obvious.

Let X, Y and U be Banach spaces, K a closed convex subset of Y and $G \colon X \times U \to Y$ a continuous function. We consider the set

$$\Phi(u) = \{x \in X \colon G(x, u) \in K\}.$$

In our case, $G(x, u) \in K$ boils down to the usual equality and inequality constraints by taking G as a function that takes values in $\mathbb{R}^{q+p}$ and $K = \{0\}^q \times (-\infty, 0]^p$.

Definition 11. Robinson's constraint qualification (RCQ) holds at a point $x_0 \in X$ such that $G(x_0, u_0) \in K$, with respect to the mapping $G(\cdot, u_0)$ and the set K, if

$$0 \in \operatorname{int}\{G(x_0, u_0) + D_x G(x_0, u_0)X - K\}.$$

Theorem 12. *Let $x_0 \in \Phi(u_0)$ be such that RCQ holds. Then, for all (x, u) in a neighbourhood of (x_0, u_0), one has*

$$\operatorname{dist}(x, \Phi(u)) = \mathcal{O}(\operatorname{dist}(G(x, u), K)).$$

This means that the geometric distance between x and the set $\Phi(u)$ on the left hand side can be controlled by the analytic distance (how much does x violate the inequality and equality constraints) on the right hand side.

When we aren't interested in pertubation analysis, i.e. we do not have the dependence from u, the constraints are given by a function $G \colon X \to Y$, so we don't have the space U. For this setting, a lot of theory on constraint qualifications has been developed. Here are some important results.

Corollary 13. *If $G \colon X \to Y$ is continuously differentiable at a point $x_0 \in \Phi := G^{-1}(K)$ and Robinson's CQ holds, then*

$$T_\Phi(x_0) := \{h \in X \colon DG(x_0)h \in T_K(G(x_0))\}. \tag{1.31}$$

This means that we can describe the tangent cone at x_0 via the derivative of G. (T_K is usually a simple object.)

Definition 14. If $G \colon X \to Y$ is continuously differentiable mapping and Φ is given by

$$\Phi = \{x \in X \colon g_i(x) = 0, i = 1, \dots, q; \quad g_i(x) \leq 0, i = q+1, \dots, p\}, \tag{1.32}$$

then Robinson's CQ is equivalent to the Mangasarian-Fromovitz (MF) CQ:

$$\begin{aligned}
&Dg_i(x_0), \quad i = 1, \dots, q, \text{ are linearly independent,} \\
&\exists h \in X \colon Dg_i(x_0)h = 0, \quad i = 1, \dots, q \\
&\qquad\qquad Dg_i(x_0)h < 0, \forall\, i \in I(x_0),
\end{aligned}$$

where $I(x_0)$ are the active inequalities at x_0.

If we strengthen the assumptions and demand that the derivatives of all equality and active inequality constraints are linearly independent, we obtain the Linear independence constraint qualification (LICQ).

Definition 15. The (generalized) Slater condition is fulfilled, if there exists a point $\bar{x} \in X$ such that $G(\bar{x}) \in \operatorname{int}(K)$.

If K is a closed, convex set with nonempty interior and G is convex with respect to $(-K)$, the Slater condition guarantees metric regularity (2.167). If in addition G is continuously differentiable, the Slater condition is equivalent to Robinson's CQ.

The simplest constraint qualification is the requirement that G is a linear function mapping into $\mathbb{R}^{q+p}$ and $K = \{0\}^q \times (-\infty, 0]^p$.

1.2.3 Optimization Problem with Perturbation Analysis

When we conduct perturbation analysis both the objective and the constraints can depend on an an additional parameter u. The goal is to assess how perturbations, that is variations of u, affect the optimal value (and solutions) of the optimization problem. Here are some useful definitions.

We consider the optimization problem

$$\min_{x \in X} f(x, u) \quad \text{subject to} \quad G(x, u) \in K, \tag{P_u}$$

where X, Y, U are Banach spaces, K is a closed convex cone of Y and f and G are continuous. The feasible set of (P_u) is given by

$$\Phi(u) = \{x \in X \colon G(x, u) \in K\}.$$

The optimal value and the set of solutions are given by

$$\nu(u) := \inf_{x \in \Phi(u)} f(x, u), \qquad \mathcal{S}(u) := \operatorname*{argmin}_{x \in \Phi(u)} f(x, u).$$

The dual problem is given by

$$\max_{\lambda \in Y^*} \inf_{x \in X} L(x, \lambda, u) - \sigma(\lambda, K), \qquad L(x, \lambda, u) = f(x, u) + \langle \lambda, G(x, u) \rangle. \tag{D_u}$$

Without any assumptions there is a duality between the primal and dual problem, that is $\operatorname{val}(D_u) \leq \operatorname{val}(P_u)$. Moreover, if we change the maximum and infimum in (D_u), we get back (P_u). The set of Lagrange multipliers is given by

$$\Lambda(x_0, u_0) := \left\{\lambda \in Y^* \colon D_x L(x_0, \lambda, u_0) = 0,\, G(x_0, u_0) \in K,\, \lambda \in K^-,\, \langle \lambda, G(x_0, u_0) \rangle = 0\right\}$$

In the common case of finitely many equality and inequality constraints the definition above turns into

$$\begin{aligned}\Lambda(x_0, u_0) := \big\{\lambda \in \mathbb{R}^{q+p} \colon \nabla_x L(x_0, \lambda, u_0) = 0,\\ g_i(x_0, u_0) = 0,\ i = 1, \dots, q;\ g_i(x_0, u_0) \leq 0,\ i = q+1, \dots, q+p,\\ \lambda_i \geq 0,\ i = q+1, \dots, q+p,\\ \lambda_i g_i(x_0, u_0) = 0,\ i = q+1, \dots, q+p\big\}\end{aligned}$$

In this case, we can linearize the problem around a point $x_0 \in X$, picking also a u_0 and a direction d in which we approach u_0. These are the linearized problem

$$\min_{h \in X} Df(x_0, u_0)(h, d) \quad \text{subject to} \quad DG(x_0, u_0)(h, d) \in T_K(G(x_0, y_0)), \tag{PL_d}$$

and its dual

$$\max_{\lambda \in \Lambda(x_0, u_0)} D_u L(x_0, \lambda, u_0) d. \tag{DL_d}$$

1.2.4 Continuity

As we saw previously, continuity is necessary to use the continuous mapping theorem to prove consistency of the optimal value.

Establishing continuity of ν seems kind of easy, when f and G are continuous. Yet, even in very simple cases, this is not true.

To do so, we can however rely on the following:

Proposition 16. *Let $u_0 \in U$ and suppose that*

(i) f is continuous on $X \times U$,

(ii) inf-compactness: There exists $\alpha \in \mathbb{R}$ and a compact set $C \subset X$ such that for all u in a neighbourhood of u_0 the level set

$$\text{lev}_\alpha f(\cdot, u) := \{x \in \Phi(u) \colon f(x, u) \leq \alpha\}$$

is non-empty and contained in C,

(iii) For all neighbourhoods V_X of $\mathcal{S}(u_0)$ there exists a neighbourhood V_U of u_0 such that $V_X \cap \Phi(u) \neq \emptyset$ for all $u \in V_U$,

then ν is continuous at u_0.

Assumption (i) is kind of natural. Assumption (ii) states that f is "well-behaved", that is for u close to u_0, the solutions will be contained in a compact set. Assumption (iii) is about the "well-behavedness" of the constraint set $\Phi(u)$: If we wiggle u, the solutions for u_0 still need to be close to $\Phi(u)$. If Robinson's CQ holds for all (x_0, u_0) with $x_0 \in \mathcal{S}(u_0)$, then Assumption (iii) follows.

1.2.5 Differentiability

(Hadamard) differentiability, on the other hand, is an hypothesis to use the Delta method in order to prove consistency. As for continuity, establishing differentiability should be easy if f and G are continuously differentiable. Again this is not the case, unfortunately.

Example 1. Take $f(x, u) = xu$ and $\Phi = [-1, 1]$. Then, $\nu(u) = -|u|$ which is clearly not differentiable at 0.

Fixed Feasible Set

First, we only look at the case where the constraint set doesn't depend on u, i.e. $\Phi(u) = \Phi$ or equivalently $G(x, u) = G(x)$.

Theorem 17. *Let $u_0 \in U$ and $d \in U$ a direction. Suppose that f is continuous on $X \times U$ and inf-compactness holds. If in addition*

(i) $f(x, \cdot)$ is (Gâteaux) differentiable for all $x \in X$ and $D_u f(x, u)$ is continuous on $X \times U$, then ν is Fréchet directionally differentiable and

$$v'(u_0, d) = \inf_{x \in \mathcal{S}(u_0)} D_u f(x, u_0) d.$$

(ii) $f(x, \cdot)$ is concave for all $x \in X$, then ν is Hadamard (quite sure about this) directionally differentiable and

$$v'(u_0, d) = \inf_{x \in \mathcal{S}(u_0)} f'_x(u_0, d).$$

Variable Feasible Set

In this section, we assume throughout that f and G are continuously differentiable.

Remark 18. A point $x \in \Phi(u)$ is an ε-optimal solution if, $f(x, u) \leq \text{val}(P_u) + \varepsilon$. This notion gives us some more generality because sometimes a (0-optimal) solution may not exist but there is an ε-optimal solution.

Robinson's CQ at $x_0 \in \Phi(u_0)$ implies directional regularity at x_0 in direction $d \in U$ for all directions d. This in turn implies Robinson's CQ for the linearized problem (PL_d).

It is relatively easy to establish an upper bound on the directional Hadamard derivative (if it exists).

Proposition 19. *Let $u_0 \in U$ and assume directional regularity in direction $d \in U$ holds for all $x_0 \in \mathcal{S}(u_0)$. Then, the upper Hadamard directional derivative is bounded as follows*

$$v'_+(u_0, d) \leq \inf_{x \in \mathcal{S}(u_0)} \text{val}(PL_d) = \inf_{x \in \mathcal{S}(u_0)} \sup_{\lambda \in \Lambda(x, u_0)} D_u L(x_0, \lambda, u_0) d.$$

Establishing a lower bound and directional Hadamard differentiability is considerably harder. Here are the three main results in the book; note the slightly varying assumptions.

Theorem 20. *Let $u_0 \in U$ and $d \in U$. Suppose*

(i) The problem (P_{u_0}) is convex and $\mathcal{S}(u_0) \neq \emptyset$,

(ii) directional regularity in direction d for all $x_0 \in \mathcal{S}(u_0)$,

(iii) For $u_n := u_0 + t_n d + o(t_n)$, where $t_n \downarrow 0$, (P_{u_n}) has an $o(t_n)$-optimal solution x_n with a subsequence x_{n_k} such that $x_{n_k} \to x_0 \in \mathcal{S}(u_0)$.

Then, the optimal value is Hadamard directionally differentiable at u_0 in direction d, and

$$\nu'(u_0, d) = \inf_{x \in \mathcal{S}(u_0)} \sup_{\lambda \in \Lambda(u_0)} D_u L(x, \lambda, u_0) d.$$

Theorem 21. *Let $u_0 \in U$ and $d \in U$. Suppose* *(i)* $\mathcal{S}(u_0) \neq \emptyset$,

(ii) directional regularity in direction d *for all* $x_0 \in \mathcal{S}(u_0)$,

(iii) For $u_n := u_0 + t_n d + o(t_n)$*, where* $t_n \downarrow 0$*,* (P_{u_n}) *has an* $o(t_n)$*-optimal solution* $\bar{x}_n$ *such that* $\operatorname{dist}(\bar{x}_n, \mathcal{S}(u_0)) = \mathcal{O}(t_n)$ *and it has a subsequence* $\bar{x}_{n_k}$ *such that* $\bar{x}_{n_k} \to x_0 \in \mathcal{S}(u_0)$.

Then, the optimal value is Hadamard directionally differentiable at u_0 *in direction* d*, and*

$$\nu'(u_0, d) = \inf_{x \in \mathcal{S}(u_0)} \sup_{\lambda \in \Lambda(x, u_0)} D_u L(x, \lambda, u_0) d.$$

Theorem 22. *Let* $u_0 \in U$ *and* $d \in U$*. Suppose*

(i) Suppose Robinson's CQ holds for all $x_0 \in \mathcal{S}(u_0)$,

(ii) For $u_n := u_0 + t_n d + o(t_n)$*, where* $t_n \downarrow 0$*,* (P_{u_n}) *has an* $o(t_n)$*-optimal solution* x_n *with a subsequence* x_{n_k} *such that* $x_{n_k} \to x_0 \in \mathcal{S}(u_0)$.

Then, for every direction d

$$\inf_{x \in \mathcal{S}(u_0)} \inf_{\lambda \in \Lambda(x, u_0)} D_u L(x, \lambda, u_0) d \;\leq\; \nu'_-(u_0, d) \;\leq\; \nu'_+(u_0, d) \;\leq\; \inf_{x \in \mathcal{S}(u_0)} \sup_{\lambda \in \Lambda(x, u_0)} D_u L(x, \lambda, u_0) d.$$

So, there are basically 3 approaches how we can guarantee directional Hadamard differentiability: having a convex problem as in Theorem 20, making sure that the solutions converge quickly enough (that is linearly) as in Theorem 21, having unique Lagrange multipliers and thus coinciding infima and suprema in Theorem 22.

Chapter 2

Methodology and Results

In this section we try to put altogether the results from the previous chapter with our work. We tried to address consistency of the optimization problem 1.26 which arises in the sensitivity analysis framework. To do so, based on the results of section 1.2, we wanted to prove under which conditions does MFCQ hold and use the equivalence with RCQ in order to apply the continuity and differentiability theorems on the optimal value function β and thus guarantee consistency.

2.1 R-value constraints

We will first of all consider constraints of the form

$$R_{X_i \sim X_j | X_{I \setminus \{i,j\}}} - B^u \leq 0, \quad -R_{X_i \sim X_j | X_{I \setminus \{i,j\}}} + B^l \leq 0. \tag{2.1}$$

Obviously, if $B^l < B^u$, only one of these constraints can be active and to prove MFCQ (or even LICQ), we need to study only active constraints.

We will express the upper bound constraints $R_{X_i \sim X_j | X_{I \setminus \{i,j\}}} - B^u \leq 0$ with $g^I_{ij}(\psi)$ and use the formulation with the covariance matrix as in equation 1.21. To establish MFCQ, we need to compute the derivative of $g^I_{ij}(\psi)$. As an intermediate step, we derive $\nabla_\psi \Omega^I_{ij}$:

$$\frac{\partial \Omega^I_{ij}}{\partial \psi_k} = \mathbf{1}\{k \in I\} \cdot \begin{cases} -(\Omega^I_{ip}\Omega^I_{kj} + \Omega^I_{ik}\Omega^I_{pj}) & \text{if } k \neq p, \\ -\Omega^I_{ip}\Omega^I_{pj} & \text{if } k = p. \end{cases}$$

One of the key steps in showing the result above is the identity

$$D_x[A^{-1}(x)] = -A^{-1}(x)(D_x[A(x)])A^{-1}(x).$$

Using this, for $k \neq p$ we obtain

$$\frac{\partial g^I_{ij}}{\partial \psi_k} = \mathbf{1}\{k \in I\} \frac{1}{\sqrt{\Omega^I_{ii}\,\Omega^I_{jj}}} \left[\Omega^I_{ip}\Omega^I_{kj} + \Omega^I_{ik}\Omega^I_{pj} - \Omega^I_{ij} \left(\frac{\Omega^I_{pj}\Omega^I_{kj}}{\Omega^I_{jj}} + \frac{\Omega^I_{pi}\Omega^I_{ki}}{\Omega^I_{ii}} \right) \right]$$

and for $k = p$ we get

$$\frac{\partial g^I_{ij}}{\partial \psi_p} = \mathbf{1}\{p \in I\} \frac{1}{2\sqrt{\Omega^I_{ii}\,\Omega^I_{jj}}} \left[2\Omega^I_{ip}\Omega^I_{pj} - \Omega^I_{ij} \left(\frac{(\Omega^I_{pj})^2}{\Omega^I_{jj}} + \frac{(\Omega^I_{ip})^2}{\Omega^I_{ii}} \right) \right].$$

Let's consider the special case where we put constraints on the sensitivity parameters

$$\tilde{g}_1 := g_{1p}^{\{1,\dots,p\}} = R_{X_1 \sim X_p | X_{\{2,\dots,p-1\}}}, \quad \tilde{g}_2 := g_{2p}^{\{2,\dots,p\}} = R_{X_2 \sim X_p | X_{\{3,\dots,p-1\}}}, \quad \text{and so on.}$$

The gradient of $\tilde{g}_l$ has the form

$$\nabla_\psi \tilde{g}_l = \begin{pmatrix} 0_{l-1} \\ + \\ *_{p-l}. \end{pmatrix}$$

Where $+$ stands for a positive entry and $*$ for an unkown real entry. This follows from using the general formula for the derivatives above and assume that the covariance matrix Σ is positive defined, which allows us to use Cauchy-Schwarz as:

$$\Omega_{ii}^I \Omega_{jj}^I - (\Omega_{ij}^I)^2 > 0 \tag{2.2}$$

Applying it to the case in study we obtain in particular that

$$\nabla_\psi g_{1p}^{\{1,\dots,p\}} = \begin{pmatrix} a_1 > 0 \\ a_2 \\ a_3 \\ a_4 \\ \dots \\ a_{p-1} \\ a_p \end{pmatrix} \quad \nabla_\psi g_{2p}^{\{2,\dots,p\}} = \begin{pmatrix} 0 \\ b_2 > 0 \\ b_3 \\ b_4 \\ \dots \\ b_{p-1} \\ b_p \end{pmatrix} \quad \nabla_\psi g_{3p}^{\{3,\dots,p\}} = \begin{pmatrix} 0 \\ 0 \\ c_3 > 0 \\ c_4 \\ \dots \\ c_{p-1} \\ c_p \end{pmatrix} \tag{2.3}$$

The gradients correspond respectively to the upper bound constraints on $R_{Y\sim U|D,Z,X}$, $R_{D\sim U|Z,X}$ and $R_{Z\sim U|X}$; if we consider the lower bound constraints we need to change signs of all the elements in the gradient.

For this case we thus have linear independence of the gradient. This is the definition LICQ, which implies MFCQ.

We note that in this case we only considered upper bound constraint but the argument does not change if for some (of all) of the above constraint we have that the active constraint is the lower bound one, e.g. $R_{Y\sim U|D,Z,X} = B^l$. In case of lower bounds the constraint is preceded by a $-$ as in 2.1, so the positive entry in the gradient becomes negative.

We can also note some further proprieties on the entries of the gradients in 2.3. In particular,by calculating the derivative we obtain that if we consider upper bound constraints:

- a_p has the opposite sign as $R_{Y\sim U|D,Z,X}$;
- b_p has the opposite sign as $R_{D\sim U|Z,X}$;
- c_p has the opposite sign as $R_{Z\sim U|X}$.

Obviously, if we consider active lower bound constraints the relation is opposite.

However, since we are interested in active constraints, we know the sign of the partial R-values once we know if the lower or upper bound is active. In particular, we note that if the upper bound is active, the partial R-value is positive, so the p-th entry is negative, while if the lower bound is active, the partial R-value is negative, and thus the p-th entry is still negative.

We can conclude that in any case, both for active upper or lower bound, a_p, b_p and c_p are negative.

2.1.1 $R_{Y\sim Z|D,X,U}$ constraint

We now would like to study the case where there is an active constraint on $R_{Y\sim Z|D,X,U}$.

We study the gradient with respect to ψ, which we denote as

$$\nabla_\psi g_{13}^{\{1,\dots,p\}} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \dots \\ d_{p-1} \\ d_p \end{pmatrix}$$

Computing the derivatives similarly as the previous section, we obtain that:

- d_1 has the opposite sign as $R_{Z\sim U|Y,D,X}$;
- d_3 has the opposite sign as $R_{Y\sim U|D,Z,X}$.

Note that, if we consider the upper bound constraints as active, d_1 has the same sign as c_p and d_3 has the same sign as a_p. If we consider some lower bound active constraints, we may change the signs opportunely.

We would like to find out whether if adding an active constraint on $R_{Y\sim Z|D,X,U}$, still guarantees some constraint qualification, at least under certain hypothesis. Note that if we have active constraints on $R_{D\sim U|Z,X}$, $R_{Z\sim U|X}$ and $R_{Y\sim Z|D,X,U}$ LICQ still holds, since we have that $d_1 \neq 0$ if $R_{Z\sim U|Y,D,X} \neq 0$.

The case in which we have all four active constraints is more tricky; trying to prove LICQ seems not to be promising, since we would have four gradients and not much information on their entries, in particular we would know that only 3 entries are equal to 0. Indeed we can impose linear independence of the gradient vectors, e.g. by imposing that $\nabla_\psi g_{13}^{\{1,\dots,p\}}$ cannot be written in terms of the first three gradients and we obtain that we need:

$$d_p \neq \frac{c_p}{c_3} d_3 - \frac{b_3 c_p}{b_2 c_3} d_2 + \frac{c_2 b_p}{b_2 c_3} d_2 - \frac{a_2 b_3}{a_1 b_2} d_1 + \frac{a_2 b_p}{a_1 b_2} d_1 + \frac{a_3 c_p}{c_3} - a_p$$

Since this condition is of course impractical, we aim to study MFCQ. Essentially the problem boils down to prove whether a solution $h = (h_1, h_2, h_3, h_4)^t$ exists for the following system of inequalities:

$$\begin{cases} a_1 h_1 + a_2 h_2 + a_3 h_3 + a_p h_p < 0 \\ b_2 h_2 + b_3 h_3 + b_p h_p < 0 \\ c_3 h_3 + c_p h_p < 0 \\ d_1 h_1 + d_2 h_2 + d_3 h_3 + d_p h_p < 0 \end{cases}$$

We will firstly address the case of all upper bound constraints; other cases are analogous and we will refer to them later. For the previous observations we have that:

- $a_1 > 0, b_2 > 0, c_3 > 0$;
- $a_p < 0, b_p < 0, c_p < 0$;
- $d_1 < 0, d_3 < 0$.

To build a solution for the system of inequalities we could proceed this way:

1. We first choose h_p such that

$$h_p > 0 \text{ if } d_p < 0, \quad h_p < 0 \text{ if } d_p > 0, \quad h_p = 0 \text{ if } d_p = 0.$$

2. We then take h_3 such that

$$h_3 < -\frac{c_p}{c_3} h_p$$

   So that the third inequality stands.

3. We choose h_2

$$h_2 < -\frac{1}{b_2}(b_3 h_3 + b_p h_p)$$

   So that the second inequality is satisfied.

4. Then, we choose h_1 such that

$$h_1 < -\frac{1}{a_1}(a_2 h_2 + a_3 h_3 + a_p h_p)$$

   So that the first inequality is satisfied.

If $d_p < 0$, we can then choose h_p such that

$$h_p > -\frac{1}{d_p}(d_1 h_1 + d_2 h_2 + d_3 h_3)$$

In this case, if we can take h_p big enough and the fourth inequality is satisfied.

If $d_p > 0$, on the other hand, it follows from the fourth inequality that we need h_p such that

$$h_p < -\frac{1}{d_p}(d_1 h_1 + d_2 h_2 + d_3 h_3)$$

In this case, to guarantee that we could choose an h_p which satisfies that condition, we would need to check that $-\frac{1}{d_p}(d_1 h_1 + d_2 h_2 + d_3 h_3)$ is not lower than h_p due to the conditions imposed to choose h_1 and h_2. In particular, when $d_p > 0$ and $d_2 < 0$, h_1, h_2, h_3 can only be chosen negative and the condition on h_p does not seem to always be possible.

We can, then, study different cases of active constraints, i.e. when we have also active lower bounds. In these cases the signs of the gradient vectors change and that could affect the existence of the solution. In particular, a relevant parameter to check is whether a_1 and d_1 have same or different sign.

If a_1 and d_1 have the same sign a solution for the system of inequality exist: if both a_1 and d_1 are positive, we can take h_1 negative enough to satisfy both the inequalities, while if the parameters are both negative, we simply take h_1 positive enough.

When $a_1 d_1 < 0$, on the other hand, a situation similar as the one of all active upper bounds analyzed before, may occur.

In table 2.1.1, we reported all the possible cases depending on the signs of $R_{Y\sim U|D,Z,X}$, $R_{Z\sim U|Y,D,X}$ and $R_{Y\sim Z|D,X,U}$. We note from the table that there are several cases where $a_1 d_1 > 0$ and thus MFCQ still holds even if all four R-value constraints are active.

$R_{Y\sim U\|D,Z,X}$	a_1	$R_{Z\sim U\|Y,D,X}$	$R_{Y\sim Z\|D,X,U}$	d_1	a_1d_1
+	> 0	+	+	< 0	−
−	< 0	+	+	< 0	+
+	> 0	−	+	> 0	+
+	> 0	+	−	> 0	+
+	> 0	−	−	< 0	−
−	< 0	+	−	> 0	−
−	< 0	−	+	> 0	−
−	< 0	−	−	< 0	+

2.1.2 Three active constraints combinations

We may want to study combinations of three active constraints. We already saw that we can prove LICQ if constraints on $R_{Y\sim U|D,Z,X}$, $R_{D\sim U|X,U}$ and $R_{Z\sim U|X}$ are active and in the case of active constraints $R_{Y\sim Z|D,X,U}$, $R_{D\sim U|X,U}$ and $R_{Z\sim U|X}$.

We would like to dive a bit deeper in other constraint combinations, in particular if we take active upper bound constraints for $R_{Y\sim U|D,Z,X}$, $R_{Z\sim U|X}$ and $R_{Y\sim Z|D,X,U}$. In this case, we have the following vectors:

$$\begin{pmatrix} a_1 > 0 \\ a_2 \\ a_3 \\ \dots \\ a_p < 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ c_3 > 0 \\ \dots \\ c_p < 0 \end{pmatrix} \quad \begin{pmatrix} d_1 < 0 \\ d_2 \\ d_3 < 0 \\ \dots \\ d_p \end{pmatrix}$$

Other constraint choices with the same R-values, i.e. if lower bound constraints are active, lead to similar situations.

We can try to study MFCQ in this case. The argument is similar to the one used in the section before to study MFCQ for the four R-value active constraints. The problem boils down also in this case to prove if a solution for a system of inequalities exists, namely:

$$\begin{cases} a_1h_1 + a_2h_2 + a_3h_3 < 0 \\ c_3h_3 < 0 \\ d_1h_1 + d_2h_2 + d_3h_3 < 0 \end{cases}$$

Analyzing this system, it follows naturally from the coefficients' signs that we need to choose h_3 negative in order to satisfy the second inequality.

To have the third equation satisfied, we can then choose h_1 such that

$$h_1 > -\frac{1}{d_1}(d_2h_2 + d_3h_3)$$

Exploiting the fact that $a_1 > 0$, it follows that we need to choose h_2 such that

$$-\frac{a_1}{d_1}(d_2h_2 + d_3h_3) < a_1h_1 < -\frac{1}{d_1}(d_2h_2 + d_3h_3)$$

which it means that

$$-\frac{a_1}{d_1}(d_2h_2 + d_3h_3) < -\frac{1}{d_1}(d_2h_2 + d_3h_3)$$

It leads to the following cases:

- if $\frac{a_1}{d_1}d_2 - a_2 > 0$, then
$$h_2 > \frac{a_3 - \frac{a_1}{d_1}d_3}{\frac{a_1}{d_1}d_2 - a_2}h_3$$
- if $\frac{a_1}{d_1}d_2 - a_2 > 0$, then
$$h_2 < \frac{a_3 - \frac{a_1}{d_1}d_3}{\frac{a_1}{d_1}d_2 - a_2}h_3$$
- if $\frac{a_1}{d_1}d_2 - a_2 = 0$, then we need
$$a_3 - \frac{a_1}{d_1}d_3 = 0$$

The first two cases can guarantee us MFCQ, since we can choose h_2 arbitrarily, depending on h_3 which can be taken arbitrarily negative.

Other constraint choices with the same R-values, i.e. if lower bound constraints are active, require analogous analysis and lead to similar results.

2.2 R^2-value constraint

2.2.1 Comparison constraints

In this section we will study constraints the constraints that involve R^2-values. We first of all list all the inequalities we consider of this form:

i $R^2_{D\sim U|\tilde{X},\dot{X}_I,Z} \leq b_{UD} R^2_{D\sim \dot{X}_J|\tilde{X},\dot{X}_I,Z}$,

ii $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z} \leq b_{UY} R^2_{Y\sim \dot{X}_J|\tilde{X},\dot{X}_I,Z}$,

iii $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z,D} \leq b_{UY} R^2_{Y\sim \dot{X}_J|\tilde{X},\dot{X}_I,Z,D}$,

iv $R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}} \leq b_{UZ} R^2_{Z\sim \dot{X}_j|\tilde{X},\dot{X}_{-j}}$,

v $R^2_{Y\sim Z|X,U,D} \leq b_{ZY} R^2_{Y\sim \dot{X}_j|\tilde{X},\dot{X}_{-j},Z,U,D}$.

We note that in the cases $(i)-(iv)$ the right-hand side depends only on observed variables $V = (Y, D, X, Z)$; in particular, it's constant with respect to ψ. It means that computing the gradient of the constraint with respect to ψ boils down to compute the gradient of the left hand side. Moreover, since on the left hand side we have a R^2-value of the form $R^2_{S\sim T|Q}$ with S and T random scalars, we can use the fact that $R^2_{S\sim T|Q} = (R_{S\sim T|Q})^2$. This relation allows us to compute the gradient of the R^2-value easily as derivative of a squared function.

To sum up, the gradients of the constraints $(i)-(iv)$ are the following:

i $2R_{D\sim U|\tilde{X},\dot{X}_I,Z}\nabla_\psi g_{2p}^{\{2,3,p\}\cup\tilde{I}\cup I}$

ii $2R_{Y\sim U|\tilde{X},\dot{X}_I,Z}\nabla_\psi g_{1p}^{\{1,3,p\}\cup\tilde{I}\cup I}$

iii $2R_{Y\sim U|\tilde{X},\dot{X}_I,Z,D}\nabla_\psi g_{1p}^{\{1,2,3,p\}\cup\tilde{I}\cup I}$

iv $2R_{Z\sim U|\tilde{X},\dot{X}_{-j}}\nabla_\psi g_{3p}^{\{3,p\}\cup\tilde{I}\cup\dot{I}\setminus\{j\}}$

where $\tilde{I}$ contains all the indexes corresponding to the Xs in $\tilde{X}$, and $\dot{I}$ the ones in $\dot{X}$. Note that for I we considered the indexes in $\dot{X}_I$

We can follow the observations from the previous sections to compute the form of the gradient vectors.

$$\nabla_\psi g_{2p}^{\{2,3,p\}\cup\tilde{I}\cup I} = \begin{pmatrix} 0 \\ + \\ * \in \mathbb{R} \\ *_{p-4} \\ - \end{pmatrix}$$

where $*_{p-4}$ has entries equal to 0 in the positions corresponding to indexes in $\dot{I}\setminus I$.

Analogously, for the other gradients:

$$\nabla_\psi g_{1p}^{\{1,3,p\}\cup\tilde{I}\cup I} = \begin{pmatrix} + \\ 0 \\ * \in \mathbb{R} \\ *_{p-4} \\ - \end{pmatrix} \quad \nabla_\psi g_{1p}^{\{1,2,3,p\}\cup\tilde{I}\cup I} = \begin{pmatrix} + \\ * \in \mathbb{R} \\ * \in \mathbb{R} \\ *_{p-4} \\ - \end{pmatrix} \quad \nabla_\psi g_{3p}^{\{3,\dots,p\}\setminus\{j\}} = \begin{pmatrix} 0 \\ 0 \\ + \\ *_{p-4} \\ - \end{pmatrix} \tag{2.4}$$

Note that in the last case the values in $*_{p-4}$ are 0 for the index j and a real value otherwise; in the first two cases, its the same as $\nabla_\psi g_{2p}^{\{2,3,p\}\cup\tilde{I}\cup I}$, so we have 0 in the positions corresponding to indexes in $\dot{I}\setminus I$.

Note that we then need to multiply the gradients in ?? for the corresponding R-values in order to obtain the signs of the gradient of the R^2-values. In Table 2.1 we summarized the form of the gradient vectors for each constraint.

We can already make some general considerations on the CQ for this first three constraints. In particular, we note that if we have active constraints on $R^2_{D\sim U|\tilde{X},\dot{X}_I,Z}$, $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z}$ (or $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z,D}$) and $R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}}$, we have linear independence, and thus LICQ. The argument is the same as the one used to prove LICQ for R-value constraints.

We now try to study the constraint (v):

$$R^2_{Y\sim Z|X,U,D} \leq b_{ZY} R^2_{Y\sim\dot{X}_j|\tilde{X},\dot{X}_{-j},Z,U,D}$$

Differently from before, we have that both the right and the left hand side depend on U, so we must take both in account to compute the derivative with respect to ψ.

First of all, we rewrite the constraint as follow:

$$\frac{R^2_{Y\sim Z|X,U,D}}{R^2_{Y\sim\dot{X}_j|\tilde{X},\dot{X}_{-j},Z,U,D}} - b_{ZY} \leq 0$$

And, using the covariance matrix:

$$\frac{(\Omega_{13}^{\{1,\dots,p\}})^2\Omega_{33}^{\{1,\dots,p\}}}{(\Omega_{1j}^{\{1,\dots,p\}})^2\Omega_{jj}^{\{1,\dots,p\}}} - b_{ZY} \leq 0$$

Note that, to lighten the notation, we assumed $\dot{X}_j = X_j$.

We denote

$$g_{1,3j}^{\{1,\dots,p\}} := \frac{(\Omega_{13}^{\{1,\dots,p\}})^2\Omega_{33}^{\{1,\dots,p\}}}{(\Omega_{1j}^{\{1,\dots,p\}})^2\Omega_{jj}^{\{1,\dots,p\}}} - b_{ZY}$$

The derivatives are computed as follow:

$$\begin{aligned}\frac{\partial g_{1,3j}^I}{\partial \psi_d} =& -\frac{\mathbf{1}\{d \in I\}}{(\Omega_{1j}^I)^4(\Omega_{33}^I)^2}\Omega_{13}^I\Omega_{1j}^I\cdot \\ &\cdot [2\Omega_{1j}^I\Omega_{33}^I\Omega_{jj}^I\Omega_{1p}^I\Omega_{d3}^I + 2\Omega_{1j}^I\Omega_{33}^I\Omega_{jj}^I\Omega_{1d}^I\Omega_{p3}^I + \\ &+ \Omega_{1j}^I\Omega_{33}^I\Omega_{13}^I\Omega_{jp}^I\Omega_{dj}^I + \Omega_{1j}^I\Omega_{33}^I\Omega_{13}^I\Omega_{jd}^I\Omega_{pj}^I + \\ &- 2\Omega_{13}^I\Omega_{jj}^I\Omega_{33}^I\Omega_{1p}^I\Omega_{dj}^I - 2\Omega_{13}^I\Omega_{jj}^I\Omega_{33}^I\Omega_{1d}^I\Omega_{pj}^I + \\ &- \Omega_{13}^I\Omega_{jj}^I\Omega_{1j}^I\Omega_{3p}^I\Omega_{d3}^I - \Omega_{13}^I\Omega_{jj}^I\Omega_{1j}^I\Omega_{3d}^I\Omega_{p3}^I]\end{aligned}$$

where $I = \{1, \dots, p\}$.
We can simplify this expression, for $d = 1$:

$$\begin{aligned}\frac{\partial g_{1,3j}^I}{\partial \psi_1} =& -\frac{2}{(\Omega_{1j}^I)^4(\Omega_{33}^I)^2}\Omega_{13}^I\Omega_{ij}^I\cdot \\ &\cdot [\Omega_{jj}^I\Omega_{1j}^I\Omega_{3p}^I(\Omega_{11}^I\Omega_{33}^I - (\Omega_{13}^I)^2) + \\ &- \Omega_{pj}^I\Omega_{13}^I\Omega_{33}^I(\Omega_{11}^I\Omega_{jj}^I - (\Omega_{1j}^I)^2)]\end{aligned}$$

However, as we can see, we can't really infer anything about the sign of the partial derivative. This leads to a more complex analysis. In particular, we can't say much about linear independence. We may still try to say something under the assumption that at least one entry in the corresponding gradient vector is non-zero, or assuming specific signs for some entries of the gradient vector and using a similar argument to the one used to study MFCQ for $R_{Y\sim Z|X,Z,D}$ constraint in Section 2.1.1.

2.2.2 Equality constraint on R^2

In order to introduce R^2-value constraints, we may also need to add the equality constraint $R^2_{U\sim\dot{X}|\tilde{X}} = 0$, that we discussed in section 1.1.4, while introducing the constraints in the optimization problem. Since it is an equality constraint, it is always active and thus must always be taken in account for the study of constraint qualifications.

Firstly, we study the simplified case where $\dot{X}$ is equal to a single X_j, thus the constraint $g_{jp}^{\tilde{I}\cup\{j,p\}}$ becomes

$$R^2_{U\sim X_j|\tilde{X}} = 0$$

In this case, the argument is similar to the constraints studied in the previous section, so the gradient corresponds to

$$2R_{U\sim X_j|\tilde{X}}\nabla_\psi g_{jp}^{\tilde{I}\cup\{j,p\}}$$

And we have, analogously as before,

$$\nabla_\psi g_{jp}^{\tilde{I}\cup\{j,p\}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ *_{p-4} \\ - \end{pmatrix}$$

where $*_{p-4}$ are 0 for the indexes not in $\tilde{I}$, a positive value in position j and real values otherwise.

We note that, even adding this equality constraint, LICQ still holds if we have constraints on $R^2_{D\sim U|\tilde{X},\dot{X}_I,Z}$, $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z}$ (or $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z,D}$) and $R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}}$ (the $(i)-(iv)$ stated above).

The general case where $\dot{X}$ is a vector is more complex, because we can't use the relation $R^2 = (R)^2$.

We try to study directly the gradient with respect to ψ of $R^2_{U\sim\dot{X}|\tilde{X}}$ using the definition of R^2:

$$R^2_{U\sim\dot{X}|\tilde{X}} = \frac{Cov(U,\dot{X}|\tilde{X})^t Var(\dot{X}|\tilde{X})^{-1} Cov(U,\dot{X}|\tilde{X})}{Var(U|\tilde{X})}$$

In terms of covariance matrix, it is

$$R^2_{U\sim\dot{X}|\tilde{X}} = \frac{(\psi_{\dot{I}} - \Sigma_{\dot{I}\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}})^t(\Sigma_{\dot{I}\dot{I}} - \Sigma_{\dot{I}\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\Sigma_{\tilde{I}\dot{I}})^{-1}(\psi_{\dot{I}} - \Sigma_{\dot{I}\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}})}{\psi_p - \psi^t_{\tilde{I}}\Sigma_{\tilde{I}\tilde{I}}\psi_{\tilde{I}}}$$

To lighten the notation, we define

$$\mathbf{r} := \psi_{\dot{I}} - \Sigma_{\dot{I}\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}} \quad \mathbf{A} := \Sigma_{\dot{I}\dot{I}} - \Sigma_{\dot{I}\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\Sigma_{\tilde{I}\dot{I}}$$

We note that, since Σ is positive defined, also $\mathbf{A}$ is positive defined

We compute the following quantities,

$$\begin{aligned}
\frac{\partial}{\partial\psi_k} R^2_{U\sim\dot{X}|\tilde{X}} =& 0 \quad \text{for } k \in \{1,2,3\} \\
\nabla_{\psi_{\dot{I}}} R^2_{U\sim\dot{X}|\tilde{X}} =& \frac{2\mathbf{A}^{-1}\mathbf{r}}{(\psi_p - \psi^t_{\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}})^2} \\
\nabla_{\psi_{\tilde{I}}} R^2_{U\sim\dot{X}|\tilde{X}} =& -2\frac{\Sigma_{\dot{I}\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\mathbf{A}^{-1}\mathbf{r}(\psi_p - \psi^t_{\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}}) + \Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}}r^t\mathbf{A}^{-1}\mathbf{r}}{(\psi_p - \psi^t_{\tilde{I}}\Sigma^{-1}_{\tilde{I}\tilde{I}}\psi_{\tilde{I}})^2} \\
\frac{\partial}{\partial\psi_p} R^2_{U\sim\dot{X}|\tilde{X}} =& -\frac{\mathbf{r}^t\mathbf{A}^{-1}\mathbf{r}}{(\psi_p - \psi^t_{\tilde{I}}\Sigma_{\tilde{I}\tilde{I}}\psi_{\tilde{I}})^2}
\end{aligned}$$

Since $\mathbf{A}$ is positive defined, we have that $\partial/\partial\psi_p(R^2_{U\sim\dot{X}|\tilde{X}}) < 0$.

We note that if we have constraints on the first three variables, even adding the equality constraint on $R^2_{U\sim\dot{X}|\tilde{X}} = 0$ with $|\dot{X}| > 1$, LICQ still holds. In particular, LICQ holds if we have a combination of

- One active constraint among $R_{Y\sim U|D,Z,X}$, $R_{Y\sim Z|D,X,U}$, $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z}$, or $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z,D}$;
- One active constraint among $R_{D\sim U|Z,X}$ or $R^2_{D\sim U|\tilde{X},\dot{X}_I,Z}$;

- One active constraint among $R_{Z\sim U|X}$ or $R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}}$.
- The equality constraint $R^2_{U\sim\dot{X}|\tilde{X}} = 0$.

2.3 Constraint combinations

In this last section we would like to summarize more precisely for which constraint combinations constraint qualifications still hold and look at some last problems.

We summarize all the constraints we could have for our problem and their gradient form in Table 2.1. In the table we used the notation that $*$ and $\dots$ correspond to element in $\mathbb{R}$ that can be positive, negative or equal to zero.

Constraint	Gradient
$R_{Y\sim U\|X,Z,D} \geq B_l^{UY}$	$\begin{pmatrix} - & * & * & \dots & - \end{pmatrix}^t$
$R_{Y\sim U\|X,Z,D} \leq B_u^{UY}$	$\begin{pmatrix} + & * & * & \dots & - \end{pmatrix}^t$
$R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UY} R^2_{Y\sim\dot{X}_J\|\tilde{X},\dot{X}_I,Z}$	$R_{Y\sim U\|\tilde{X},\dot{X}_I,Z} \cdot \begin{pmatrix} + & 0 & * & \dots & - \end{pmatrix}^t$ with 0 for the indexes of X which are not in $\dot{X}_I$
$R^2_{Y\sim U\|\tilde{X},\dot{X}_I,Z,D} \leq b_{UY} R^2_{Y\sim\dot{X}_J\|\tilde{X},\dot{X}_I,Z,D}$	$R_{Y\sim U\|\tilde{X},\dot{X}_I,Z,D} \cdot \begin{pmatrix} + & * & * & \dots & - \end{pmatrix}^t$ with 0 for the indexes of X which are not in $\dot{X}_I$
$R_{D\sim U\|X,Z} \geq B_l^{UD}$	$\begin{pmatrix} 0 & - & * & \dots & - \end{pmatrix}^t$
$R_{D\sim U\|X,Z} \leq B_u^{UD}$	$\begin{pmatrix} 0 & + & * & \dots & - \end{pmatrix}^t$
$R^2_{D\sim U\|\tilde{X},\dot{X}_I,Z} \leq b_{UD} R^2_{D\sim\dot{X}_J\|\tilde{X},\dot{X}_I,Z}$	$R_{D\sim U\|\tilde{X},\dot{X}_I,Z} \cdot \begin{pmatrix} 0 & + & * & \dots & - \end{pmatrix}^t$ with 0 for the indexes of X which are not in $\dot{X}_I$
$R_{Z\sim U\|X} \geq B_l^{UZ}$	$\begin{pmatrix} 0 & 0 & - & \dots & - \end{pmatrix}^t$
$R_{Z\sim U\|X} \leq B_u^{UZ}$	$\begin{pmatrix} 0 & 0 & + & \dots & - \end{pmatrix}^t$
$R^2_{Z\sim U\|\tilde{X},\dot{X}_{-j}} \leq b_{UZ} R^2_{Z\sim\dot{X}_j\|\tilde{X},\dot{X}_{-j}}$	$R_{Z\sim U\|\tilde{X},\dot{X}_{-j}} \cdot \begin{pmatrix} 0 & 0 & + & \dots & - \end{pmatrix}^t$ with 0 in position j
$R_{Y\sim Z\|X,U,D} \geq B_l^{ZY}$	$\begin{pmatrix} \text{sign}(R_{Z\sim U\|X,D,Y}) & * & \text{sign}(R_{Y\sim U\|X,Z,D}) & \dots & * \end{pmatrix}^t$
$R_{Y\sim Z\|X,U,D} \leq B_u^{ZY}$	$\begin{pmatrix} -\text{sign}(R_{Z\sim U\|X,D,Y}) & * & -\text{sign}(R_{Y\sim U\|X,Z,D}) & \dots & * \end{pmatrix}^t$
$R^2_{Y\sim Z\|X,U,D} \leq b_{ZY} R^2_{Y\sim\dot{X}_j\|\tilde{X},\dot{X}_{-j},Z,U,D}$	$\begin{pmatrix} * & * & * & \dots & * \end{pmatrix}^t$
$R^2_{U\sim\dot{X}\|\tilde{X}} = 0$	$\begin{pmatrix} 0 & 0 & 0 & \dots & - \end{pmatrix}^t$

Table 2.1: Summary of constraints and associated gradients

We notice that in general we can say something about the signs of the first three and the last vector entries, but we can't say much about the signs of the elements corresponding to the variables in X, i.e. the indexes $(4, \dots, p-1)$. It follows that it seems promising to study constraint qualifications only when we have no more than 4 active constraints; in particular, we have several combinations with three active constraints involving the first three variables Y, D, Z with U and the equality constraint on $R^2_{U\sim\dot{X}|\tilde{X}}$, for which we can prove LICQ directly looking at the signs of vector entries. In particular, every combination of active constraints with:

- One active constraint among the ones involving $R_{Y\sim U|D,Z,X}$, $R_{Y\sim Z|D,X,U}$, $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z}$, or $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z,D}$;

- One active constraint among the ones involving $R_{D\sim U|Z,X}$ or $R^2_{D\sim U|\tilde{X},\dot{X}_I,Z}$;
- One active constraint among the one involving $R_{Z\sim U|X}$ or $R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}}$;
- The constraint $R^2_{U\sim\dot{X}|\tilde{X}}$.

If we want to add further active constraints we may look for MFCQ rather than LICQ. We studied MFCQ in Section 2.1.1 and we have that MFCQ holds in the combinations with:

- One active constraint among $R_{Y\sim U|D,Z,X}$, $R_{Y\sim Z|D,X,U}$, $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z}$, or $R^2_{Y\sim U|\tilde{X},\dot{X}_I,Z,D}$;
- One active constraint among $R_{D\sim U|Z,X}$ or $R^2_{D\sim U|\tilde{X},\dot{X}_I,Z}$;
- One active constraint among $R_{Z\sim U|X}$ or $R^2_{Z\sim U|\tilde{X},\dot{X}_{-j}}$;
- The constraint $R_{Y\sim Z|X,U,D}$.

where at least one of the active constraints is on R^2-values.

We saw that in case that all four constraints are R-values, we have MFCQ under specific conditions. Moreover, MFCQ holds if three out of the four constraints are active.

In the MFCQ case, however, it would be necessary to study the compatibility of $R^2_{U\sim\dot{X}|\tilde{X}}$ with the R^2-value constraints. If we denote as

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \alpha_p \end{pmatrix}$$

the gradient vector of the equality constraint on $R^2_{U\sim\dot{X}|\tilde{X}}$. To still have MFCQ we need that $\alpha_p h_p = 0$, so $h_p = 0$. However, taking $h_p = 0$ could lead to problems to have the system of inequalities satisfied for the other constraints and needs to be studied more in detail.

2.3.1 Positive definiteness constraint

In all the previous sections, we assumed that the covariance matrix Σ is positive defined and extensively used this hypothesis to prove the signs of gradient vectors through Cauchy-Schwarz inequalities.

However, it would be necessary to include the positive definiteness of the covariance matrix as a constraint of the optimization problem and thus include it in our analysis of constraint qualifications.

The block matrix

$$M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

is positive definite iff $A \succ 0$ and $M \setminus A = C - B^T A^{-1} B \succ 0$. Applying this to our covariance matrix $\Sigma(\psi)$, we find that the requirement for positive definiteness translates to one additional constraint

$$h(\psi) = \psi_p - \psi_J^T (\Sigma_{JJ})^{-1} \psi_J > 0,$$

where $J = \{1, \ldots, p-1\}$. This follows because we already know that Σ_{JJ} is positive definite and does not depend on ψ. The gradient of the additional inequality constraint h is given by

$$\nabla_{\psi} h(\psi) = \begin{pmatrix} -2(\Sigma_{JJ})^{-1}\psi_J \\ 1 \end{pmatrix}.$$

Hence, the signs of the different components depend on both θ and ψ and thus we cannot deduce much for our problem.

Conclusion

This report has examined the consistency properties of the optimal values of stochastic optimization problems in the context of optimization-based sensitivity analysis for unmeasured confounding. Building on the framework developed by Freidling and Zhao in [2], we investigated the theoretical foundations of this approach, which aims to quantify the sensitivity of causal effect estimates by bounding the effects using stochastic optimization.

Our work focused on identifying conditions under which the optimal values of these problems are consistent with the data-derived estimators. We reviewed existing theoretical results, developed a methodology to assess consistency, and provided sufficient conditions to guarantee it under specific scenarios. Central to our analysis was the role of constraint qualifications, such as the Mangasarian-Fromovitz conditions, which ensure the feasibility and well-posedness of the optimization problems. The conditions we derived are detailed and theoretically robust, offering a foundation for future studies. However, the study also highlighted a key limitation: the conditions we found must be evaluated at the optimal solution of the stochastic optimization problem, which is generally unknown. This dependency reduces the practicality of the results and underscores the need for further refinements to the framework.

Future research directions could aim to address this limitation by exploring methods to ensure that the required constraint qualifications hold almost surely under the Lebesgue measure, thereby excluding critical edge cases where consistency might fail.

This study opens avenues for a more general exploration of consistency in stochastic optimization, trying to better understand conditions for the Hadamard differentiability of the optimal value function. Potential research could include developing examples and counterexamples that illustrate interesting behavior of the optimal value function under varying conditions, examining the effects of near-violations of constraint qualifications, and refining the theoretical conditions under which sensitivity analysis remains robust. Moreover, a stochastic simulation study could be conducted to evaluate the empirical properties of the optimal value's convergence, offering practical insights to complement the theoretical findings.

In summary, this work provides a first step toward a broader theory of consistency in stochastic optimization problems, with a particular focus on their application to sensitivity analysis. While significant progress has been made, addressing the highlighted limitations and pursuing the outlined research directions can lead to a deeper understanding of the interplay between statistical estimation and optimization. This would ultimately contribute to the development of more robust and widely applicable sensitivity analysis methodologies.

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